

KIRWAN SURJECTIVITY FOR QUIVER VARIETIES

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ABSTRACT. For algebraic varieties defined by hyperkähler or, more generally, algebraic symplectic reduction, it is a long-standing question whether the “hyperkähler Kirwan map” on cohomology is surjective. We resolve this question in the affirmative for Nakajima quiver varieties. We also establish similar results for other cohomology theories and for the derived category. Our proofs use only classical topological and geometric arguments.

1. INTRODUCTION

Suppose M is a complex algebraic variety with the action of a complex algebraic group G , yielding a quotient stack/equivariant space $\mathfrak{X} = M/G$; or more generally \mathfrak{X} is any complex algebraic stack. Often \mathfrak{X} has one or more natural open sets \mathfrak{X}^{ss} —typically defined via geometric invariant theory (GIT)—that are smooth algebraic varieties; thus, when $\mathfrak{X} = M/G$, we have $\mathfrak{X}^{\text{ss}} = M^{\text{ss}}/G$ where G acts freely on M^{ss} . Fixing such an open subset $i : \mathfrak{X}^{\text{ss}} \hookrightarrow \mathfrak{X}$, one has the following problem.

Kirwan Surjectivity Problem. *When is the pullback map*

$$(1.1) \quad H^*(\mathfrak{X}) \xrightarrow{i^*} H^*(\mathfrak{X}^{\text{ss}})$$

surjective?

Convention 1.1. Except when noted otherwise, H^* means singular cohomology with \mathbb{Z} coefficients.

When \mathfrak{X} itself is smooth and \mathfrak{X}^{ss} is defined by GIT, classical Morse-theoretic results of Atiyah-Bott and Kirwan show that the “Kirwan map” (1.1) is surjective. Significant recent attention focuses on the case when \mathfrak{X} is a singular, but algebraic symplectic or even hyperkähler, stack: typically, letting Z be a smooth G -variety with algebraic moment map $\mu : T^*Z \rightarrow \mathfrak{g}^* = \text{Lie}(G)^*$, we have $\mathfrak{X} = \mu^{-1}(0)/G$ and $\mathfrak{X}^{\text{ss}} = \mu^{-1}(0)^{\text{ss}}/G$ for a choice of GIT stability.

This paper resolves the Kirwan Surjectivity Problem when \mathfrak{X}^{ss} is a Nakajima quiver variety.

Thus, let $Q = (I, \Omega)$ be a quiver and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_{\geq 0}^I$ vectors with $\mathbf{w} \neq \mathbf{0}$. Following Nakajima [Nak1, Nak2], these data yield (notation as in Section 3.1):

- (1) a finite-dimensional complex vector space $\mathbb{M} = \mathbb{M}(\mathbf{v}, \mathbf{w})$, with
- (2) the linear action of the complex group $\mathbb{G} = \prod_i GL_{v_i}$, and
- (3) a (complex) moment map $\mu : \mathbb{M} \rightarrow \text{Lie}(\mathbb{G})^*$.

Fixing the character $\chi : \mathbb{G} \rightarrow \mathbb{G}_m$ defined by $\chi(\prod_i g_i) = \prod_i \det(g_i)^{-1}$ yields a stability condition in the sense of GIT, with stable locus $\mu^{-1}(0)^{\text{ss}} = \mu^{-1}(0)^s \subset \mathbb{M}^s$. The \mathbb{G} -action on \mathbb{M}^s is free, and the quotient $\mathfrak{M} = \mathfrak{M}(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0)^s/\mathbb{G}$ is the Nakajima quiver variety associated to Q and \mathbf{v}, \mathbf{w} .

Theorem 1.2. *Let $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ be a smooth Nakajima quiver variety. Then the Kirwan map*

$$H_{\mathbb{G}}^*(\text{pt}) \cong H_{\mathbb{G}}^*(\mu^{-1}(0)) \longrightarrow H_{\mathbb{G}}^*(\mu^{-1}(0)^s) = H^*(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$$

is surjective. Thus, $H^(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$ is generated by tautological classes.*

We note that $H_{\mathbb{G}}^*(\text{pt})$ is a polynomial ring (in the tautological classes of the theorem). Theorem 1.2 extends to many other cohomology theories, including complex K -theory and elliptic cohomology:

Theorem 1.3. *Assume that $E^*(\mathrm{pt})$ is concentrated in even degrees.*

- (1) *The map $E^*(B\mathbb{G}) \rightarrow E^*(\mathfrak{M})$ is surjective.*
- (2) *If E is complex-oriented, then $E^*(\mathfrak{M})$ is generated as an $E^*(\mathrm{pt})$ -algebra by Chern classes of tautological bundles.*

Corollary 1.4. *The natural maps $K_{\mathbb{G}}^*(\mathrm{pt}) \rightarrow K^*(\mathfrak{M})$ and $\mathrm{Ell}_{\mathbb{G}}^*(\mathrm{pt}) \rightarrow \mathrm{Ell}^*(\mathfrak{M})$ are surjective.*

Furthermore, if \mathbb{T} is an algebraic torus acting on \mathfrak{M} , then, because $H^*(\mathfrak{M})$ is evenly graded, the Leray spectral sequence for $H_{\mathbb{T}}^*(\mathfrak{M})$ degenerates, showing that $H^*(\mathfrak{M}) = H_{\mathbb{T}}^*(\mathfrak{M}) \otimes_{H_{\mathbb{T}}^*(\mathrm{pt})} H^*(\mathrm{pt})$. Via Theorem 1.2 and the Nakayama Lemma for graded rings, we conclude:

Corollary 1.5. *For a torus \mathbb{T} acting on \mathfrak{M} , the map $H_{\mathbb{G} \times \mathbb{T}}^*(\mathrm{pt}) \rightarrow H_{\mathbb{T}}^*(\mathfrak{M})$ is surjective.*

In particular, the expectation expressed in Section 2.2.2 of [AO]—that their map (9) is an embedding near the origin of $\mathcal{E}_{\mathbb{T}}$ —follows. We note the applicability of the above results in other, similar contexts (cf. [MO]). Analogues of Corollary 1.5 can also be proven for K -theory and elliptic cohomology equivariant with respect to a torus \mathbb{T} or more general “flavor symmetries” of \mathfrak{M} .

Our method also yields the following.

Theorem 1.6. *Let $D(\mathfrak{M})$ denote the unbounded quasicoherent derived category of \mathfrak{M} , and $D_{\mathrm{coh}}(\mathfrak{M})$ its bounded coherent subcategory.*

- (1) *The category $D(\mathfrak{M})$ is generated by tautological bundles.*
- (2) *There is a finite list of tautological bundles from which every object of $D_{\mathrm{coh}}(\mathfrak{M})$ is obtained by finitely many applications of (i) direct sum, (ii) cohomological shift, and (iii) cone.*

We note that the second assertion of Theorem 1.6 is *not* simply a formal consequence of the first, since we do *not* include taking direct summands (i.e., retracts) among the operations (i)–(iii).

We mention one further application of Theorem 1.2 (that will be readily apparent to experts).

Corollary 1.7 (Assumption 5.13 of [BDMN]). *Let $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$, and $Z := Z(\mathfrak{g})^* \subset \mathfrak{g}^*$ denote the dual of the center. Consider the family $\mathfrak{M} = \mu^{-1}(Z)^{\mathrm{ss}}/\mathbb{G} \rightarrow Z$ of Hamiltonian reductions. Then the Duistermaat-Heckman map for this family is surjective. In particular, the family of Hamiltonian reductions $\mathfrak{M} \rightarrow Z$ provides a versal Poisson deformation of the Nakajima quiver variety \mathfrak{M} .*

Cases of Kirwan surjectivity for quivers of finite and affine Dynkin type have previously been established [V, SVV, W] by different techniques.

Here is a sketch of the strategy used to prove Theorem 1.2.

- (1) We compactify \mathfrak{M} to a projective variety $\overline{\mathfrak{M}}$ by an explicit quiver construction.
- (2) We identify the class of the graph Γ of the inclusion $i : \mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$ in $H^*(\mathfrak{M} \times \overline{\mathfrak{M}})$ as a Chern class of a complex built from external tensor products of tautological bundles on $\mathfrak{M} \times \overline{\mathfrak{M}}$.
- (3) Purely topological arguments allow us to conclude (Section 2) that the Chern classes of the tautological bundles generate the cohomology of \mathfrak{M} .

Hiding behind our approach to Theorem 1.2 and our other results is a general pattern (that experts may already discern here) for moduli spaces in noncommutative geometry—that is, moduli of objects in certain categories. The general story will be worked out in a forthcoming paper [McGN]. Nonetheless, it seemed desirable to us to present the results for quiver varieties separately. Indeed, on the one hand, the proof of Theorem 1.2 can be made completely classical and explicit for quivers, in a way that avoids any categorical yoga or abstraction (and thus will be of independent interest to some readers). On the other hand, we also obtain sharper results for quiver varieties than seem to be easily achievable in a completely general context.

Convention 1.8. Throughout the paper, all varieties, groups, etc. are defined over \mathbb{C} .

Acknowledgments. We thank G. Bellamy for comments on a draft. The first author was supported by EPSRC programme grant EI/I033343/1. The second author was supported by NSF grants DMS-1159468 and DMS-1502125.

2. TOPOLOGY OF COMPACTIFICATIONS

Throughout the paper, we use $H^*(X)$, with no further decorations indicating coefficients, to denote cohomology with \mathbb{Z} -coefficients, and $H_*^{\text{BM}}(X)$ to denote Borel-Moore homology with \mathbb{Z} -coefficients; if X is smooth, there is a canonical isomorphism $H^*(X) \cong H_*^{\text{BM}}(X)$.

2.1. Pushforwards and the Projection Formula. Suppose $f : X \rightarrow Y$ is a proper morphism of relative dimension d of smooth, connected varieties (or Deligne-Mumford stacks). Then there is a pushforward, or Gysin, map $f_* : H^*(X) \rightarrow H^{*-d}(Y)$.

The Gysin map satisfies the projection formula: for classes $c \in H^*(X)$, $c' \in H^*(Y)$, we have

$$(2.1) \quad f_*(c \cup f^*c') = f_*(c) \cup c'.$$

Moreover, if $f : X \rightarrow Y$ is a closed immersion, then

$$(2.2) \quad f_*f^*c = c \cup [X],$$

where $[X]$ denotes the fundamental class of X in Borel-Moore homology (which is canonically isomorphic to cohomology since Y is smooth).

2.2. Künneth Components and Images Under Pullback. Suppose $C \in H^*(X \times Y)$ is a cohomology class. By the Künneth theorem $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$, and thus we may write

$$(2.3) \quad C = \sum x_i \otimes y_i \text{ with } x_i \in H^*(X), y_i \in H^*(Y).$$

The classes x_i, y_i are the *left-hand, respectively right-hand, Künneth components* of C with respect to the decomposition (2.3); they are not independent of the choice of decomposition (2.3).

Now suppose that $f : X \rightarrow Y$ is a morphism from a smooth variety X to a smooth, proper variety Y . Let $\Gamma_f \subset X \times Y$ be the graph of the map.

Proposition 2.1. *The image of $f^* : H^*(Y) \rightarrow H^*(X)$ is contained in the span of the Künneth components of $[\Gamma_f]$ with respect to X (and any decomposition as in (2.3)).*

Proof. Write $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$ for the projections and, abusively, $\Gamma_f : X \rightarrow X \times Y$ for both the graph immersion and its image. Write $p_* : Y \rightarrow \text{Spec}(\mathbb{C})$ for the projection to a point. Then $(p_X)_*$ exists since Y is proper, and

$$f^*d = (p_X)_*(\Gamma_f)_* \Gamma_f^* p_Y^* d = (p_X)_*([\Gamma_f] \cup p_Y^* d) = \sum_i (p_X)_*[(p_X^* x_i \cup p_Y^* y_i) \cup p_Y^* d] = \sum_i x_i \cup p_*(y_i \cup d).$$

This proves the claim. \square

2.3. Resolution of a Graph. Again suppose that $f : X \rightarrow Y$ is a morphism from a smooth variety to an irreducible projective variety, with graph $\Gamma \subset X \times Y$. We assume $f(X) \subset Y^{\text{sm}}$, the smooth locus of Y . We consider the situation in which $f^* : H^*(Y) \rightarrow H^*(X)$ is surjective.

Example 2.2. If $H_*^{\text{BM}}(X, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$ is generated by algebraic cycles and $X \rightarrow Y$ is an open immersion, then $H^*(Y, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ is surjective.

Remark 2.3. For a Nakajima quiver variety \mathfrak{M} , $H^*(\mathfrak{M}, \mathbb{Z})$ is generated by algebraic cycles by Theorem 7.3.5 of [Nak2].

Continuing with the above situation, let \tilde{Y} be a resolution of singularities; since $f(X)$ does not intersect the singular locus of Y , f lifts canonically to a morphism $\tilde{f} : X \rightarrow \tilde{Y}$ and the preimage of Γ_f in $X \times \tilde{Y}$ is $\Gamma_{\tilde{f}}$.

Proposition 2.4. *Suppose that*

$$(2.4) \quad R: \bigoplus_j \mathcal{E}_j^{-1} \boxtimes \mathcal{F}_j^{-1} \longrightarrow \bigoplus_j \mathcal{E}_j^0 \boxtimes \mathcal{F}_j^0 \longrightarrow \bigoplus_j \mathcal{E}_j^1 \boxtimes \mathcal{F}_j^1$$

is a complex of vector bundles on $X \times Y$ with the following properties.

- (1) $\mathcal{H}^1(R) = 0$, $\mathcal{H}^{-1}(R) = 0$, and $\mathcal{H} := \mathcal{H}^0(R)$ *is a vector bundle on $X \times Y$.*
- (2) $\mathrm{rk}(\mathcal{H}) = d := \dim(Y)$.
- (3) $s \in H^0(X \times Y, \mathcal{H})$ *is a section with zero locus $Z(s) = \Gamma$.*

Letting $\tilde{Y} \rightarrow Y$ be a resolution of singularities, write

$$\tilde{R}: \bigoplus_j \mathcal{E}_j^{-1} \boxtimes \tilde{\mathcal{F}}_j^{-1} \longrightarrow \bigoplus_j \mathcal{E}_j^0 \boxtimes \tilde{\mathcal{F}}_j^0 \longrightarrow \bigoplus_j \mathcal{E}_j^1 \boxtimes \tilde{\mathcal{F}}_j^1$$

for the pullback of R to $X \times \tilde{Y}$ and $\tilde{\mathcal{H}} = \mathcal{H}^0(\tilde{R})$. Then:

- (i) $c_d(\tilde{\mathcal{H}}) = [\Gamma_{\tilde{Y}}]$ *in $X \times \tilde{Y}$.*
- (ii) *The Chern classes of $\tilde{\mathcal{H}}$ are polynomials, with integer coefficients, in the Chern classes of the bundles \mathcal{E}_j^ℓ and $\tilde{\mathcal{F}}_j^\ell$.*
- (iii) *The image of the map $H^*(\tilde{Y}, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ is contained in the span of the Chern classes of the bundles \mathcal{E}_j^ℓ .*

Proof. (i) It is standard that if the zero locus of a section of a vector bundle $\tilde{\mathcal{H}}$ of rank d has codimension d —in which case it is a local complete intersection subscheme—then its fundamental class equals $c_d(\tilde{\mathcal{H}})$.

(ii) By the additivity of Chern classes, we have

$$c(\tilde{\mathcal{H}}) = \prod_j c(\mathcal{E}_j^0 \boxtimes \tilde{\mathcal{F}}_j^0) \prod_j c(\mathcal{E}_j^{-1} \boxtimes \tilde{\mathcal{F}}_j^{-1})^{-1} \prod_j c(\mathcal{E}_j^1 \boxtimes \tilde{\mathcal{F}}_j^1)^{-1}.$$

The inverses of the total Chern classes are the total Segre classes, which are known to be polynomials, with integer coefficients, in the Chern classes: see Chapter 5 of [Ful]. Moreover, the Chern classes of $\mathcal{E}_j^\ell \boxtimes \tilde{\mathcal{F}}_j^\ell$ are also polynomials (with integer coefficients) in the Chern classes of \mathcal{E}_j^ℓ and $\tilde{\mathcal{F}}_j^\ell$: see Example 14.5.2 of [Ful].¹

(iii) By parts (i) and (ii), the class $[\Gamma_{\tilde{Y}}]$ has a Künneth decomposition (2.3) whose left-hand components are integer polynomials in the Chern classes of the bundles \mathcal{E}_j^ℓ . Assertion (iii) is now immediate from Proposition 2.1. \square

Corollary 2.5. *Suppose that \mathfrak{M} is a smooth Nakajima quiver variety and $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$ is an open immersion in a projective variety. If the graph Γ of the immersion can be written as the zero locus $Z(s)$ of a section $s \in H^0(\mathfrak{M} \times \overline{\mathfrak{M}}, \mathcal{H})$ of a vector bundle \mathcal{H} as in Proposition 2.4, then $H^*(\mathfrak{M}, \mathbb{Z})$ is generated by the Chern classes of the bundles \mathcal{E}_j^ℓ .*

Proof. As explained above, $H^*(\mathfrak{M}, \mathbb{Z})$ is known to be generated by algebraic cycles; hence (cf. Proposition 1.8 of [Ful]) for any projective compactification $\overline{\mathfrak{M}}$ the restriction map $H^*(\overline{\mathfrak{M}}, \mathbb{Z}) \rightarrow H^*(\mathfrak{M}, \mathbb{Z})$ is surjective. The assertion is now immediate from Proposition 2.4. \square

¹This is, however, abstractly clear: the Chern classes are pulled back along the composite $X \times \tilde{Y} \rightarrow BGL(\mathrm{rk}(\mathcal{E}_j^\ell)) \times BGL(\mathrm{rk}(\tilde{\mathcal{F}}_j^\ell)) \xrightarrow{\otimes} BGL(\mathrm{rk}(\mathcal{E}_j^\ell) \cdot \mathrm{rk}(\tilde{\mathcal{F}}_j^\ell))$, hence are polynomials in the cohomology classes generating $H^*(BGL(\mathrm{rk}(\mathcal{E}_j^\ell)) \times BGL(\mathrm{rk}(\tilde{\mathcal{F}}_j^\ell)))$.

3. QUIVER VARIETIES

3.1. Basics of Quivers. Let (I, E) be an undirected graph with vertex set I and edge set E . Following Nakajima [Nak1, Nak2], we let H denote the set of pairs of an edge with an orientation; thus H comes with source and target maps $s, t : H \rightarrow I$. Given $h \in H$, we let \bar{h} denote the same edge with opposite orientation, so $s(\bar{h}) = t(h)$ and $t(\bar{h}) = s(h)$.

Next, fix a preferred orientation for each edge: in other words, fix a decomposition $H = \Omega \sqcup \bar{\Omega}$ where $\bar{\Omega} = \{\bar{h} \mid h \in \Omega\}$. We let $Q = (I, \Omega)$ denote the quiver, i.e., the finite directed graph, with vertices I and arrows Ω ; then $Q^{\text{dbl}} = (I, H)$ is the associated *doubled quiver*. We define a function

$$\epsilon : H \longrightarrow \{\pm 1\} \quad \text{by} \quad \epsilon(h) = \begin{cases} 1 & \text{if } h \in \Omega, \\ -1 & \text{if } h \in \bar{\Omega}. \end{cases}$$

The *preprojective algebra* is the quotient $\Pi^0(Q) = kQ^{\text{dbl}} / \left(\sum_{h \in H} \epsilon(h) \bar{h} h \right)$ of the path algebra kQ^{dbl} of the doubled quiver. The relation $\sum_{h \in H} \epsilon(h) \bar{h} h = 0$ is the *preprojective relation*.

If V_\bullet is an I -graded vector space, then $\text{Rep}(Q, V_\bullet) = \bigoplus_{h \in \Omega} \text{Hom}(V_{s(h)}, V_{t(h)})$. When $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$ and $V_i = \mathbb{C}^{v_i}$ for all $i \in I$, we write $\text{Rep}(Q, \mathbf{v}) = \text{Rep}(Q, V_\bullet)$.

Let $\mathbf{v} = (v_i)_{i \in I}$, $\mathbf{w} = (w_i)_{i \in I}$ be dimension vectors, and $V_i, W_i (i \in I)$ be complex vector spaces with $\dim(V_i) = v_i, \dim(W_i) = w_i$; here W_i are the *framing vector spaces*. Given pairs $\mathbf{v}^1, \mathbf{w}^1$ and $\mathbf{v}^2, \mathbf{w}^2$ and vector spaces $V_i^j, W_i^j (j = 1, 2)$ as above, let

$$L(V^1, V^2) = \bigoplus_{i \in I} \text{Hom}(V_i^1, V_i^2), \quad E(V^1, V^2) = \bigoplus_{h \in H} \text{Hom}(V_{s(h)}^1, V_{t(h)}^2).$$

One has obvious actions of $L(V^1, V^2)$ on $L(V^2, V^3)$, $L(V^3, V^1)$, $E(V^2, V^3)$, and $E(V^3, V^1)$.

On p. 520 of [Nak1], Nakajima defines a bilinear multiplication

$$E(V^2, V^3) \times E(V^1, V^2) \rightarrow L(V^1, V^3), \quad \text{by}$$

$$(C, B) \mapsto CB = \left(\sum_{t(h)=k} C_h B_{\bar{h}} \right)_k \in L(V^1, V^3).$$

Now, fixing \mathbf{v}, \mathbf{w} and collections of vector spaces $(V_i), (W_i)$ as above, let

$$\mathbb{M} = \mathbb{M}(\mathbf{v}, \mathbf{w}) = E(V, V) \oplus L(W, V) \oplus L(V, W).$$

We write $[B, i, j]$ for an element of \mathbb{M} . The group

$$\mathbb{G} = \mathbb{G}(\mathbf{v}) = \prod_i GL(V_i) \cong \prod_i GL_{v_i}$$

acts linearly on \mathbb{M} in the obvious way. There is a canonical moment map, coming from the identification of \mathbb{M} as a cotangent bundle to a linear space, written $\mu : \mathbb{M} \longrightarrow \text{Lie}(\mathbb{G})^*$.

3.2. Crawley-Boevey's Construction. Suppose $Q = (I, \Omega)$ is a quiver with dimension vectors \mathbf{v}, \mathbf{w} as above. To such data, Crawley-Boevey associates [CB, Section 1] a new quiver, that we will denote by Q^{CB} . It has vertex set $I^{\text{CB}} = I \cup \{\infty\}$, and oriented arrows

$$\Omega^{\text{CB}} = \Omega \cup \{a_{(i,j)} \mid s(a_{(i,j)}) = \infty, t(a_{(i,j)}) = i, i \in I, j \in \{1, \dots, \mathbf{w}_i\}\}.$$

In other words, we add \mathbf{w}_i -many arrows from ∞ to i . Let $\alpha \in \mathbb{Z}_{\geq 0}^{I^{\text{CB}}}$ be the dimension vector for Q^{CB} that equals \mathbf{v}_i at $i \in I$ and 1 at the vertex ∞ . Then $\mathbb{M}(\mathbf{v}, \mathbf{w}) = T^* \text{Rep}(Q^{\text{CB}}, \alpha)$. Also the

natural homomorphism $\mathbb{G} \rightarrow G(\alpha) := \prod_{i \in I^{\text{CB}}} GL(\alpha_i)/\mathbb{G}_m$ (where \mathbb{G}_m is the diagonal multiplicative group) is an isomorphism, making the identification of $\mathbb{M}(\mathbf{v}, \mathbf{w})$ with $T^* \text{Rep}(Q^{\text{CB}}, \alpha)$ equivariant. It is immediate that the two canonical moment maps coincide.

3.3. Semistability and Stability for Quiver Representations. Fix a quiver $Q = (I, \Omega)$ with dimension vector \mathbf{v} and write $\mathbb{G} = \prod_{i \in I} GL(\mathbf{v}_i)$ as above. Recalling the moment map

$$\mu : \mathbb{M}(\mathbf{v}, \mathbf{w}) \rightarrow \text{Lie}(\mathbb{G})^*$$

above, the Nakajima quiver variety associated to dimension vector \mathbf{v} and framing vector \mathbf{w} is

$$\mathfrak{M} = \mathfrak{M}(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0) //_{\chi_0} \mathbb{G}.$$

Here the quotient is the GIT quotient with respect to the character $\chi_0 : \mathbb{G} \rightarrow \mathbb{G}_m$ defined by $\chi_0(g_i) = \prod_i \det(g_i)^{-1}$. It is known [Nak1] that with respect to this character, semistability and stability coincide on \mathbb{M} .

Given a character $\chi_0 : \mathbb{G} \rightarrow \mathbb{G}_m$, let $\delta : \mathbb{G}_m \rightarrow \mathbb{G}$ be the diagonal \mathbb{G}_m and write $\chi_0(\delta(z)) = z^d$. Then we get a character $\chi : \mathbb{G} \times \mathbb{G}_m \rightarrow \mathbb{G}_m$ by $\chi(g, z) = \chi_0(g)z^{-d}$. Then χ is trivial on the diagonal \mathbb{G}_m in $\mathbb{G} \times \mathbb{G}_m$, and thus it factors through a character of $G(\alpha)$, which obviously agrees with χ_0 under the isomorphism $\mathbb{G} \rightarrow G(\alpha)$.

Proposition 3.1 (p. 251 of [CB]). *With respect to the character $\chi_0(g_i) = \prod_i \det(g_i)^{-1}$, the stable and semistable loci of $T^* \text{Rep}(Q^{\text{CB}}, \alpha)$ coincide, and consist of those representations of the double of Q^{CB} for which every nonzero sub-representation is nonzero at the vertex ∞ .*

A representation for which every nonzero subrepresentation is nonzero at ∞ is said to be *cogenerated at ∞* [CB, p. 251].

3.4. Tautological Bundles and Nakajima's Section. We continue with a quiver $Q = (I, \Omega)$. Let V^1 and V^2 be I -graded vector spaces of dimension \mathbf{v} , and W an I -graded vector space of dimension \mathbf{w} . One defines functors from \mathbb{G} -representations, respectively $\mathbb{G} \times \mathbb{G}$ -representations, to \mathbb{G} -equivariant vector bundles on a \mathbb{G} -variety Z , respectively to $\mathbb{G} \times \mathbb{G}$ -equivariant vector bundles on a $\mathbb{G} \times \mathbb{G}$ -variety, by $R \mapsto \mathcal{R} := \mathcal{O} \otimes_{\mathbb{C}} R$.

In particular, each V_i^j defines a \mathbb{G} -equivariant vector bundle \mathcal{V}_i^j on \mathbb{M} , and the $\mathbb{G} \times \mathbb{G}$ -representations $L(V^1, V^2), E(V^1, V^2), L(W, V^2), L(V^1, W)$ define $\mathbb{G} \times \mathbb{G}$ -equivariant vector bundles

$$\mathcal{L}(V^1, V^2), \quad \mathcal{E}(V^1, V^2), \quad \mathcal{L}(W, V^2), \quad \mathcal{L}(V^1, W)$$

on $\mathbb{M} \times \mathbb{M}$ (where $\mathbb{G} \times \mathbb{G}$ acts on V^1 via the first factor and on V^2 via the second factor).

Remark 3.2. In the language of stacks, these bundles are pullbacks along $\mathbb{M}/\mathbb{G} \times \mathbb{M}/\mathbb{G} \rightarrow B\mathbb{G} \times B\mathbb{G}$.

Nakajima defines $\mathbb{G} \times \mathbb{G}$ -equivariant homomorphisms,

$$(3.1) \quad \mathcal{L}(V^1, V^2) \xrightarrow{\sigma} \mathcal{E}(V^1, V^2) \oplus \mathcal{L}(W, V^2) \oplus \mathcal{L}(V^1, W) \xrightarrow{\tau} \mathcal{L}(V^1, V^2),$$

where at a point $([B, i, j], [B', i', j']) \in \mathbb{M} \times \mathbb{M}$ the maps σ, τ are given by

$$(3.2) \quad \sigma(\xi) = (B'\xi - \xi B, -\xi i, j'\xi), \quad \tau(C, a, b) = \epsilon B'C + \epsilon C B + i'b + aj.$$

Proposition 3.3 ([Nak1], p. 537 and Lemma 5.2). *Suppose $[B, i, j], [B', i', j'] \in \mathbb{M}$.*

- (1) *If $[B', i', j'] \in \mathbb{M}^s$ then σ is injective in the fiber over $([B, i, j], [B', i', j'])$.*
- (2) *If $[B, i, j] \in \mathbb{M}^s$ then τ is surjective in the fiber over $([B, i, j], [B', i', j'])$.*
- (3) *$\tau \circ \sigma = 0$ in the fiber over $([B, i, j], [B', i', j']) \in \mu^{-1}(0) \times \mu^{-1}(0)$.*

Now define a section s of $\mathcal{E}(V^1, V^2) \oplus \mathcal{L}(W, V^2) \oplus \mathcal{L}(V^1, W)$ by

$$(3.3) \quad s([B, i, j], [B', i', j']) = (0, -i', j).$$

Proposition 3.4 ([Nak1], p. 537).

- (1) Over $\mathbb{M} \times \mathbb{M}$, we have $\tau(s) = 0$.
- (2) Viewing $s|_{\mu^{-1}(0) \times \mu^{-1}(0)^s}$ as a section of $\text{coker}(\sigma)$, its vanishing locus $Z(s)$ in $\mu^{-1}(0)^s \times \mu^{-1}(0)^s$ is smooth and equals the locus of pairs $([B, i, j], [B', i', j'])$ for which

$$\mathbb{G} \cdot [B, i, j] = \mathbb{G} \cdot [B', i', j'].$$

We now want to translate the above in terms of the Crawley-Boevey quiver Q^{CB} . Consider framed representations $[B, i, j], [B', i', j'] \in \mu^{-1}(0)^s \times \mu^{-1}(0)^s$, acting on the vector spaces (V^1, W) and (V^2, W) (both with associated dimensions \mathbf{v}, \mathbf{w}).

We write $B^{\text{CB}}, (B')^{\text{CB}}$ for the associated representations of the preprojective algebra $\Pi^0(Q^{\text{CB}})$, and $(V^\ell)^{\text{CB}}$ for their underlying vector spaces. Thus, one has

$$(V^\ell)_j^{\text{CB}} = \begin{cases} V_i^\ell & \text{if } j = i \in I; \\ \mathbb{C} & \text{if } j = \infty. \end{cases}$$

Now

$$(3.4) \quad L((V^1)^{\text{CB}}, (V^2)^{\text{CB}}) = L(V^1, V^2) \oplus \text{Hom}(\mathbb{C}, \mathbb{C}),$$

$$(3.5) \quad E((V^1)^{\text{CB}}, (V^2)^{\text{CB}}) = E(V^1, V^2) \oplus L(W, V^2) \oplus L(V^1, W).$$

The following is immediate from (3.4), (3.5), and Proposition 3.4:

Proposition 3.5.

- (1) Under the identifications of (3.4), (3.5), the map

$$L((V^1)^{\text{CB}}, (V^2)^{\text{CB}}) = L(V^1, V^2) \oplus \mathbb{C} \xrightarrow{\sigma \oplus s} E(V^1, V^2) \oplus L(W, V^2) \oplus L(V^1, W) = E((V^1)^{\text{CB}}, (V^2)^{\text{CB}})$$

is identified with the map

$$\partial_0 : L((V^1)^{\text{CB}}, (V^2)^{\text{CB}}) \longrightarrow E((V^1)^{\text{CB}}, (V^2)^{\text{CB}})$$

defined by $\partial_0(\phi) = (B')^{\text{CB}}\phi - \phi B^{\text{CB}}$.

- (2) Thus, for the dual map

$$\partial_0^\vee : \mathcal{E}((V^1)^{\text{CB}}, (V^2)^{\text{CB}}) \longrightarrow \mathcal{L}((V^1)^{\text{CB}}, (V^2)^{\text{CB}})$$

we have that $\text{coker}(\partial_0^\vee)$ is the direct image to $\mu^{-1}(0)^s \times \mu^{-1}(0)^s$ of a line bundle on the smooth subvariety of part (2) of Proposition 3.4.

4. GRADED TRIPLED QUIVERS AND THEIR MODULI SPACES

The present section is intended to provide a compactification of the moduli space of representations of the preprojective algebra $\Pi^0(Q)$ associated to a quiver Q . For applications to Nakajima quiver varieties associated to a quiver Q_0 , set $Q = Q_0^{\text{CB}}$, the Crawley-Boevey quiver associated to Q_0 .

4.1. Graded Tripling of a Quiver. Let (I, E) be a graph, $\alpha \in \mathbb{Z}_{\geq 0}^I$ a dimension vector for I . Fix an orientation Ω defining a quiver $Q = (I, \Omega)$ as in Section 3.1. Fixing a closed interval $[a, b] \subset \mathbb{Z}$, we define a new quiver associated to (I, Ω) , the *graded-tripled quiver*, denoted Q^{gtr} , as follows. We give Q^{gtr} the vertex set $I \times [a, b]$ where I is the vertex set of Q . If E is the edge set of Q and H the associated set of pairs of an edge together with an orientation, we give Q^{gtr} the arrow set

$$(H \times [a, b - 1]) \cup (I \times [a, b - 1]).$$

Thus:

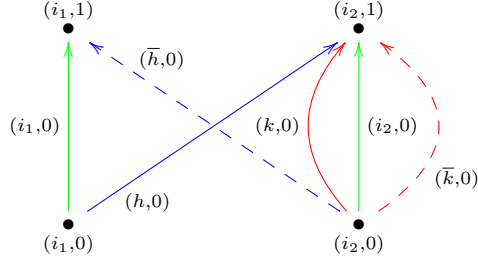
(1) for each $h \in H$, $n \in [a, b-1]$ we have arrows (h, n) with

$$s(h, n) = (s(h), n) \quad \text{and} \quad t(h, n) = (t(h), n+1);$$

(2) for each $i \in I$, $n \in [a, b-1]$ we have arrows (i, n) with

$$s(i, n) = (i, n) \quad \text{and} \quad t(i, n) = (i, n+1).$$

For example, taking $[a, b] = [0, 1]$ and Q to be $\bullet_{i_1} \xrightarrow{h} \bullet_{i_2} \xrightarrow{k} \bullet_{i_2}$ yields, as Q^{gtr} , the quiver:



Remark 4.1. Letting $b \rightarrow \infty$, the constructions extend *mutatis mutandis* to the case $[a, \infty) \subset \mathbb{Z}$.

Given a dimension vector α for Q , we define a “constant dimension vector” α^{gtr} for Q^{gtr} by $\alpha_{i,n}^{\text{gtr}} = \alpha_i$ for all $i \in I, n \in [a, b]$.

4.2. Relations and Representations. We will consider Q^{gtr} as a quiver with relations. Many of the relations are derived from those for the preprojective algebra $\Pi^0(Q)$.

We fix a decomposition $H = \Omega \sqcup \bar{\Omega}$ as in Section 3.1, determining a function ϵ .

Notation 4.2. We write:

- (1) $a_{h,n}$ for the generators of kQ^{gtr} corresponding to arrows (h, n) (where $h \in H, n \in [a, b-1]$);
- (2) $e_{i,n}$ for the generators of kQ^{gtr} corresponding to arrows (i, n) (where $i \in I, n \in [a, -1]$).

Definition 4.3. We write $A := kQ^{\text{gtr}}/I$, where I is the two-sided ideal in the path algebra kQ^{gtr} generated by the following relations:

- (1) $\sum_{h \in H} \epsilon(h) a_{\bar{h}, n+1} a_{h, n}$, $n \in [a, b-2]$ (“preprojective relations”).
- (2) $e_{t(h), n+1} a_{h, n} - a_{h, n+1} e_{s(h), n}$ for all $n \in [a, b-2]$, $h \in H$.

We note that it is immediate from condition (2) that the elements $e_n := \sum_{i \in I} e_{i, n}$ are actually central in A : all other required relations hold trivially in the path algebra of Q^{gtr} .

We write $\text{Rep}(Q^{\text{gtr}}, \alpha^{\text{gtr}})$ for the space of representations of Q^{gtr} with dimension vector α^{gtr} : thus, fixing an $I \times [a, b]$ -graded vector space $V_{\bullet, \bullet} = \bigoplus_{i \in I, n \in [a, b]} V_{i, n}$ with dimension vector α^{gtr} , we set

$$\text{Rep}(Q^{\text{gtr}}, V_{\bullet, \bullet}) = \left(\bigoplus_{h \in H, n \in [a, b-1]} \text{Hom}(V_{s(h), n}, V_{t(h), n+1}) \right) \oplus \left(\bigoplus_{i \in I, n \in [a, b-1]} \text{Hom}(V_{i, n}, V_{i, n+1}) \right).$$

We write $\text{Rep}(Q^{\text{gtr}}, \alpha^{\text{gtr}})$ when $V_{i, n} = \mathbb{C}^{\alpha_{i, n}^{\text{gtr}}}$. We also write

$$\text{Rep}(A, V_{\bullet, \bullet}) \subseteq \text{Rep}(Q^{\text{gtr}}, V_{\bullet, \bullet}), \quad \text{respectively} \quad \text{Rep}(A, \alpha^{\text{gtr}}) \subseteq \text{Rep}(Q^{\text{gtr}}, \alpha^{\text{gtr}})$$

for the closed affine subscheme of representations of A (that is, representations of Q^{gr} satisfying the relations generating I). We will write $\mathbb{G} = \prod_{i \in I} GL(\alpha_i)$ for the group associated to Q and dimension vector α ; then $\mathbb{G}^{\text{gr}} \cong \mathbb{G} \times [a, b]$ naturally acts on the affine schemes $\text{Rep}(A, \alpha^{\text{gr}}) \subseteq \text{Rep}(Q^{\text{gr}}, \alpha^{\text{gr}})$.

Remark 4.4. We note that this choice of notation is not entirely consistent with our earlier notation in the context of Nakajima quiver varieties. When $Q = Q_0^{\text{CB}}$ is the Crawley-Boevey quiver associated to Q_0 , we will write $\mathbb{G}_0 = \prod_{i \in I_0} GL(\alpha_i)$.

Consider $\Pi^0 = \Pi^0(Q)$ as a graded algebra (with all generators corresponding to arrows $h \in H$ in degree 1). Let $\Pi^0[e]$ be the graded polynomial extension with $\deg(e) = 1$.

Lemma 4.5. *Suppose $V_{\bullet, \bullet}$ is an $I \times [a, b]$ -graded vector space. Letting $h \in \Pi^0$ act via $\sum_n a_{h,n} \in \text{Rep}(Q^{\text{gr}}, V_{\bullet, \bullet})$ and e act via $\sum_{i,n} e_{i,n} \in \text{Rep}(Q^{\text{gr}}, V_{\bullet, \bullet})$, the space of graded $\Pi^0[e]$ -module structures on $V_{\bullet, \bullet}$ is naturally identified with $\text{Rep}(A, V_{\bullet, \bullet})$.*

4.3. From Π^0 -Modules to Q^{gr} -Representations. Suppose we have a finite-dimensional representation $V = (V_i)_{i \in I}$ of the preprojective algebra Π^0 of dimension vector α .

Construction 4.6. We obtain a representation of A on a vector space $V_{\bullet, \bullet}$ of dimension vector α^{gr} defined by:

- (1) setting $V_{i,n} := V_i$ for all $i \in [a, b]$;
- (2) defining each $e_{i,n} : V_{i,n} = V_i \xrightarrow{\text{id}} V_i = V_{i,n+1}$ to act by shift of \mathbb{Z} -grading; and
- (3) defining each generator of A corresponding to $h \in H$ to act via Π^0 followed by grading shift.

The construction determines a morphism of algebraic varieties (“induction”)

$$\text{Ind}^\circ : \text{Rep}(\Pi^0, V) \longrightarrow \text{Rep}(A, V_{\bullet, \bullet}).$$

$$\text{Write } \mathbb{G} = \prod_i GL(V_i) \text{ and } \mathbb{G}^{\text{gr}} = \prod_{(i,n) \in I \times [a,b]} GL(V_{i,n}) \cong \prod_{n \in [a,b]} \mathbb{G} \text{ as above,}$$

with the diagonal homomorphism $\text{diag} : \mathbb{G} \rightarrow \mathbb{G}^{\text{gr}} \cong \prod_{n \in [a,b]} \mathbb{G}$. Then the morphism Ind° is $(\mathbb{G}, \mathbb{G}^{\text{gr}})$ -equivariant. We thus get a natural \mathbb{G}^{gr} -equivariant morphism

$$(4.1) \quad \text{Ind} : \mathbb{G}^{\text{gr}} \times_{\mathbb{G}} \text{Rep}(\Pi^0, V) \longrightarrow \text{Rep}(A, V_{\bullet, \bullet}).$$

Thus, given a representation $(a_h : V_{s(h)} \rightarrow V_{t(h)})_{h \in H}$ of Π^0 on V , and $(g_{i,n}) \in \mathbb{G}^{\text{gr}}$, we have

$$\text{Ind}((g_{i,n}), a_h) = (a_{h,n}, e_{i,n}) \text{ where } a_{h,n} = g_{t(h),n+1} a_h g_{s(h),n}^{-1} \text{ and } e_{i,n} = g_{i,n+1} g_{i,n}^{-1}.$$

Proposition 4.7. *The map Ind of (4.1) defines an open immersion of $\mathbb{G}^{\text{gr}} \times_{\mathbb{G}} \text{Rep}(\Pi^0, V)$ in $\text{Rep}(A, V_{\bullet, \bullet})$, whose image consists of those $(a_{h,n}, e_{i,n})$ for which:*

$$(\dagger) \quad e_{i,n} \text{ is an isomorphism for all } n \in [a, b-1].$$

Proof. The condition (\dagger) is clearly an open condition. Given $(a_{h,n}, e_{i,n})$ satisfying (\dagger) , define

$$a_h := e_{t(h),a}^{-1} a_{h,a}, g_{i,a} := \text{Id}_i, g_{i,n} := e_{i,n-1} e_{i,n-2} \dots e_{i,a} \text{ for } n \geq a+1.$$

Inductively applying the identity $e_{t(h),a+1} a_{h,a} = a_{h,a+1} e_{s(h),a}$, one calculates that $(g_{i,n}) \cdot \text{Ind}^\circ(a_h) = (a_{h,n}, e_{i,n})$. This construction $(a_{h,n}, e_{i,n}) \mapsto ((g_{i,n}), a_h) \in \mathbb{G}^{\text{gr}} \times_{\mathbb{G}} \text{Rep}(\Pi^0, V)$ is evidently inverse to Ind on the locus of those $(a_{h,n}, e_{i,n})$ that satisfy the condition (\dagger) . \square

Corollary 4.8. *The morphism of quotient stacks*

$$\mathrm{Ind}^\circ : \mathrm{Rep}(\Pi^0, V)/\mathbb{G} \longrightarrow \mathrm{Rep}(A, V_{\bullet, \bullet})/\mathbb{G}^{\mathrm{gtr}}$$

is an open immersion.

Remark 4.9. We note that if $V_{\bullet, \bullet}$ lies in the open image of Ind , then it uniquely determines an $I \times \mathbb{Z}$ -graded $\Pi^0[e]$ -module $\tilde{V}_{\bullet, \bullet}$ with $\tilde{V}_{i, n} = \alpha_i$ for all $n \in \mathbb{Z}$ and $i \in I$. In other words, $V_{\bullet, \bullet}$ uniquely extends “upwards and downwards” to all graded degrees compatibly with the $\Pi^0[e]$ -action.

4.4. Characters and Stability. We maintain the setting of Sections 4.1 and 4.2. Fix a quiver $Q = (I, \Omega)$ and dimension vector α for Q . Let $\mathbb{G} = \prod_i GL(\alpha_i)$ denote the group determined by Q .

Following [Ki], given a character $\chi : \mathbb{G} \rightarrow \mathbb{G}_m$, write

$$\chi((g_i)_{i \in I}) = \prod_{i \in I} \det(g_i)^{\theta_i} \quad \text{and} \quad \theta = (\theta_i)_{i \in I}.$$

Given an I -graded vector space $(M_i)_{i \in I}$, we define $\delta_i(M) = \dim(M_i)$, and thus define

$$\theta = \sum_i \theta_i \delta_i \quad \text{so that} \quad \theta(M) = \sum_i \theta_i \dim(M_i).$$

Associated to χ one gets a corresponding notion of GIT semistability as in [Ki]. In particular, by Proposition 3.1 of [Ki], a representation V of Q for which $\theta(V) = 0$ is χ -semistable, respectively stable, if for every nonzero proper subrepresentation $M \subset V$, we have

$$\theta(M) \geq 0, \quad \text{respectively } \theta(M) > 0.$$

4.5. Stability for Crawley-Boevey Quivers. Suppose that $Q_0 = (I_0, \Omega_0)$ is a quiver with dimension vector \mathbf{v} and framing vector \mathbf{w} and that $Q = Q_0^{\mathrm{CB}} = (I, \Omega)$ is the associated Crawley-Boevey quiver, with dimension vector α so that $\alpha_\infty = 1$ and $\alpha|_{I_0} = \mathbf{v}$. We fix $[a, b] \subset \mathbb{Z}$ and let Q^{gtr} denote the quiver constructed above from Q . We write α^{gtr} for the associated dimension vector: thus,

$$\alpha_{i, n}^{\mathrm{gtr}} = \alpha_i = \begin{cases} \mathbf{v}_i & i \in I_0 \\ 1 & i = \infty. \end{cases}$$

We want to choose a character χ^{gtr} of $\mathbb{G}^{\mathrm{gtr}}$, or equivalently a linear functional on $\mathbb{Z}_{\geq 0}^{I \times [a, b]}$, with the following properties:

- (1) If V has dimension vector α^{gtr} , then for every nonzero proper submodule $M \subset V$ we have $\theta(M) \neq 0$. In particular, the semistable and stable points of $\mathrm{Rep}(Q^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}})$ coincide.
- (2) If V is a representation associated to a representation of the preprojective algebra $\Pi^0(Q)$, then V is stable if and only if the corresponding $\Pi^0(Q)$ -representation is stable.

We first remind the reader that $\delta_{i, n}(M) := \dim(M_{i, n})$; we will write θ as a linear combination of the $\delta_{i, n}$. Also, we note that it suffices to construct a *rational* linear functional θ , since any positive integer multiple of θ , equivalently χ , evidently defines the same stable and semistable loci.

In our construction of θ , we will want to fix positive integers

$$(4.2) \quad T \gg S := \sum_{(i, n) \in I \times [a, b]} \alpha_{i, n}^{\mathrm{gtr}} \quad \text{and} \quad N, A \gg 0.$$

We fix an ordering on the vertices of Q_0 , namely $I = \{i_1, \dots, i_r\}$. We write θ as a sum of terms:

$$\begin{aligned} \theta^0 &= \sum_{i_j \in I_0} [\delta_{i_j, b} - (1 + T^{-j})\delta_{i_j, a}], & \theta^{\mathrm{mid}} &= T^{-N} \left[\sum_{(i, n) \in I \times (a, b)} \delta_{i, n} \right], \\ \text{and} \quad \theta^\infty &= T^A \delta_{\infty, b} - [T^A + \theta^0(V) + \theta^{\mathrm{mid}}(V)] \delta_{\infty, a}. \end{aligned}$$

Finally, $\theta := \theta^0 + \theta^{\text{mid}} + \theta^\infty$. Since $\delta_{\infty,a}(V) = 1 = \delta_{\infty,b}$ we get $\theta(V) = 0$. Expanding θ in powers of T , we get:

$$(4.3) \quad \theta = T^A [\delta_{\infty,b} - \delta_{\infty,a}] + T^0 \left[-(\theta^0(V) + \theta^{\text{mid}}(V))\delta_{\infty,a} + \sum_{i_j \in I_0} (\delta_{i_j,b} - \delta_{i_j,a}) \right] \\ - \sum_{i_j \in I_0} T^{-j} \delta_{i_j,a} + T^{-N} \left[\sum_{(i,n) \in I \times (a,b)} \delta_{i,n} \right].$$

Lemma 4.10. *For fixed dimension vector α (and thus α^{gtr}) and choices as in (4.2),*

$$\theta(M) \neq 0 \quad \text{for } 0 \not\subseteq M \not\subseteq V.$$

Proof. Assume that $\theta(M) = 0$. Write $m_{i,n} := \delta_{i,n}(M) = \dim(M_{i,n})$. We note the easy estimates

$$|\theta^0(M)| \leq 2S < T, \quad 0 \leq \theta^{\text{mid}}(M) \leq 1, \quad |\theta^\infty(M)| \geq \frac{T^A}{2} \text{ if } m_{\infty,a} \neq m_{\infty,b}.$$

Thus $\theta(M) = 0$ implies $m_{\infty,a} = m_{\infty,b}$. Since $m_{\infty,b} \in \{0, 1\}$, we consider the two cases separately.

Case 1. $m_{\infty,a} = m_{\infty,b} = 0$. In this case $\theta^\infty(M) = 0$. We get

$$(4.4) \quad 0 = \theta^{\text{mid}}(M) + \theta^0(M) = \sum_{i_j \in I_0} [m_{i_j,b} - m_{i_j,a}] + \sum_{i_j \in I_0} T^{-j} m_{i_j,a} + T^{-N} \sum_{(i,n) \in I \times (a,b)} m_{i,n}.$$

Each coefficient of a power of T is much smaller than T by (4.2), so we conclude

$$m_{i,n} = 0 \quad \text{for } (i,n) \in (I \times (a,b)) \cup (I_0 \times \{a\}).$$

Thus the vanishing of the first sum in (4.4) implies $m_{i_j,b} = 0$ for all $i_j \in I_0$. Taken together with the hypothesis of Case 1, we conclude that all $m_{i,n}$ are zero, or $M = 0$.

Case 2. $m_{\infty,a} = m_{\infty,b} = 1$. Then

$$0 = \theta(M) = \theta^0(M) + \theta^{\text{mid}}(M) - \theta^0(V) - \theta^{\text{mid}}(V) \\ = \sum_{i_j \in I_0} [m_{i_j,b} - m_{i_j,a}] - \sum_{i_j \in I_0} T^{-j} [m_{i_j,a} - \alpha_{i_j,a}^{\text{gtr}}] + T^{-N} \sum_{(i,n) \in I \times (a,b)} [m_{i,n} - \alpha_{i,n}^{\text{gtr}}].$$

Each coefficient of a power of T is much smaller than T by (4.2), so we conclude

$$(4.5) \quad \sum_{i_j \in I_0} [m_{i_j,b} - m_{i_j,a}] = 0, \quad m_{i_j,a} - \alpha_{i_j,a}^{\text{gtr}} = 0 \text{ for all } j, \quad \sum_{(i,n) \in I \times (a,b)} [m_{i,n} - \alpha_{i,n}^{\text{gtr}}] = 0.$$

Since

$$m_{i,n} \leq \alpha_{i,n}^{\text{gtr}} \quad \text{for all } i, n,$$

we conclude:

- $m_{i_j,a} = \alpha_{i_j,a}^{\text{gtr}}$ for $i_j \in I_0$ by the middle equality of (4.5), hence
- $\sum_{i_j \in I_0} [m_{i_j,b} - \alpha_{i_j,b}^{\text{gtr}}] = 0$ by the left-hand equality of (4.5) using $\alpha_{i_j,a}^{\text{gtr}} = \alpha_{i_j,b}^{\text{gtr}}$, hence
- $m_{i_j,b} = \alpha_{i_j,b}^{\text{gtr}}$ for all $i_j \in I_0$, and also
- $m_{i,n} = \alpha_{i,n}^{\text{gtr}}$ for $(i,n) \in I \times (a,b)$ by the right-hand equality of (4.5).

Taken all together, we conclude that $m_{i,n} = \alpha_{i,n}^{\text{gtr}}$ for all $(i,n) \in I \times [a,b]$, i.e. $M = V$. \square

Proposition 4.11. *With respect to the θ (and thus character $\chi = \chi_\theta$) constructed above, we have:*

- (1) *The semistable and stable loci of $\text{Rep}(Q^{\text{gtr}}, \alpha^{\text{gtr}})$ coincide, as do those of $\text{Rep}(A, \alpha^{\text{gtr}})$.*
- (2) *Every stable point of $\text{Rep}(Q^{\text{gtr}}, \alpha^{\text{gtr}})$ is generated as an A -module in degree a .*

- (3) If $V_{\bullet,\bullet}, W_{\bullet,\bullet}$ are vector spaces with dimension vector α^{gtr} , equipped with A -module structures making them stable, then $\text{Hom}_A(V_{\bullet,\bullet}, W_{\bullet,\bullet})$ is 1-dimensional if $V_{\bullet,\bullet}$ and $W_{\bullet,\bullet}$ are isomorphic as A -modules and is 0-dimensional otherwise.
- (4) For a representation V of Π^0 of dimension vector α , V is stable with respect to the Crawley-Boevey character if and only if $\text{Ind}^\circ(V) \in \text{Rep}(A, \alpha^{\text{gtr}})$ is stable with respect to the character χ_θ determined by our choice of θ .

Proof. (1) This is the content of Lemma 4.10.

(2) Supposing V is stable, let M be the subrepresentation generated by $V_{I \times \{a\}}$. As in the proof of Lemma 4.10, since $\delta_{\infty,a}(M) = 1$, we have that $\theta^\infty(M) < -\frac{T^A}{2}$, and thus V is unstable, unless also $\delta_{\infty,b}(M) = 1$. Thus, we conclude that

$$\theta(M) = -\theta^0(V) - \theta^{\text{mid}}(V) + \theta^0(M) + \theta^{\text{mid}}(M).$$

It is clear that:

- $\theta^{\text{mid}}(M) - \theta^{\text{mid}}(V) \leq 0$ with equality iff $m_{i,n} = \alpha_{i,n}^{\text{gtr}}$ for all $(i,n) \in I \times (a,b)$.
- $\theta^0(M) - \theta^0(V) \leq 0$ with equality iff $m_{i_j,b} = \alpha_{i_j,b}^{\text{gtr}}$ for all $i_j \in I_0$.

Since V is assumed stable, we conclude that $m_{i,n} = \alpha_{i,n}^{\text{gtr}}$ for all $i,n \in I \times [a,b]$, as claimed.

(3) is standard.

(4) Recall (Proposition 3.1) that, for a representation V of $\Pi^0(Q)$, V is stable if and only if for any nonzero subrepresentation $M \subseteq V$ we have $M_\infty \neq 0$: that is, V is “cogenerated at the vertex ∞ .” As before, we write α^{gtr} for the dimension vector of $\text{Ind}^\circ(V)$ where V has dimension vector α .

Consider a representation V of $\Pi^0(Q)$ and a Q^{gtr} subrepresentation $M \subseteq \text{Ind}^\circ(V)$ so that $M = \text{Ind}^\circ(V')$ for a destabilizing subrepresentation $0 \neq V' \subsetneq V$. Writing $m_{i,n} = \dim(M_{i,n})$, we have

$$m_{i,n+1} = m_{i,n} \text{ for } n \in [a, b-1], \quad m_{\infty,n} = 0 \text{ for all } n.$$

Thus $\theta(M) = -\sum_{i_j \in I_0} T^{-j} m_{i_j,a} + T^{-N}(b-a-1) \sum_{(i,n) \in I \times (a,b)} m_{i,n}$. It now follows from (4.2) that

$\theta(M) < 0$ since some $m_{i_j,a} \neq 0$, so M is a witness to the instability of $\text{Ind}^\circ(V)$.

Conversely, suppose a Q^{gtr} -subrepresentation $0 \neq M \subsetneq \text{Ind}^\circ(V)$ destabilizes $\text{Ind}^\circ(V)$. Because $e_{i,n} : \text{Ind}^\circ(V)_{i,n} \rightarrow \text{Ind}^\circ(V)_{i,n+1}$ is an isomorphism for each $n \in [a, b-1]$, we have that $\dim M_{i,n+1} \geq \dim M_{i,n}$ for all $(i,n) \in I \times [a, b-1]$. Now

$$\begin{aligned} \theta(M) = T^A(m_{\infty,b} - m_{\infty,a}) + \sum_{i_j \in I_0} [m_{i_j,b} - m_{i_j,a}] - \sum_{i_j \in I_0} T^{-j} [m_{i_j,a} - m_{\infty,a} \alpha_{i_j,a}^{\text{gtr}}] \\ + T^{-N} \sum_{(i,n) \in I \times (a,b)} [m_{i,n} - m_{\infty,a} \alpha_{i,n}^{\text{gtr}}]. \end{aligned}$$

By (4.2), if $m_{\infty,b} > m_{\infty,a}$ then via the first term on the right-hand side we conclude $\theta(M) > 0$; since M destabilizes, we conclude $m_{\infty,b} = m_{\infty,a}$. Similarly, (4.2) implies that if some $m_{i_j,b} > m_{i_j,a}$ for $i_j \in I_0$ then, via the second term on the right-hand side, $\theta(M) > 0$; we conclude that $m_{i_j,b} = m_{i_j,a}$ for all $i_j \in I_0$, which together with the previous sentence implies that $m_{i,n} = m_{i,n+1}$ for all $(i,n) \in I \times [a, b-1]$. Thus $M = \text{Ind}^\circ(V')$ for a $\Pi^0(Q)$ -submodule $V' \subsetneq V$.

Now if $m_{\infty,a} = 0$ then $\theta(M) \leq 0$ implies that some $m_{i_j,a} \neq 0$, i.e. that $V' \neq 0$ but $V'_\infty = 0$, and thus V' destabilizes V . On the other hand, if $m_{\infty,a} = 1$ then $\theta(M)$ reduces to

$$\theta(M) = \sum_{i_j \in I_0} T^{-j} [\alpha_{i_j,a}^{\text{gtr}} - m_{i_j,a}] + T^{-N} \sum_{(i,n) \in I \times (a,b)} [m_{i,n} - \alpha_{i,n}^{\text{gtr}}].$$

Since we assumed $\theta(M) \leq 0$, we get $m_{i_j,a} = \alpha_{i_j,a}^{\text{gtr}}$ for all $i_j \in I_0$, which together with the conclusion of the previous paragraph implies that $M = \text{Ind}^\circ(V)$, contradicting our assumption on M . We

conclude that if $\text{Ind}^\circ(V)$ is θ -unstable then V is unstable as a $\Pi^0(Q)$ -module. This completes the proof of (4). \square

As in [Ki, Proposition 4.3], since Q^{gtr} has no oriented cycles we obtain a *projective* quotient

$$\overline{\mathfrak{M}} := \text{Rep}(A, \alpha^{\text{gtr}}) //_{\chi^{\text{gtr}}} \mathbb{G}^{\text{gtr}}.$$

Corollary 4.12. *The natural map $\text{Ind} : \mathfrak{M} \rightarrow \overline{\mathfrak{M}}$ is an open immersion of the quiver variety \mathfrak{M} in a projective scheme.*

Remark 4.13. Although it appears that $\overline{\mathfrak{M}}$ is nonsingular and connected in the instances we care about, we do not need this. Instead, we may replace $\overline{\mathfrak{M}}$ by the closure of \mathfrak{M} in $\overline{\mathfrak{M}}$ and give that closure the reduced scheme structure. Thus,

in what follows we always assume without comment that $\overline{\mathfrak{M}}$ is integral and projective.

5. A PERFECT COMPLEX ON $\mathfrak{M} \times \overline{\mathfrak{M}}$

We note that the construction in this section is similar to the one in Section 5 of [Nak1]. However, we wish to emphasize that Nakajima's framings are *not* explicitly present in this section: for applications to Nakajima quiver varieties with nonzero framing, one should take $Q = (Q_0)^{\text{CB}}$ to be the Crawley-Boevey quiver associated to the quiver Q_0 used in Nakajima's constructions.

Fix a quiver Q and a dimension vector α . Let $V_{\bullet, \bullet}, W_{\bullet, \bullet}$ be two $I \times [a, b]$ -graded vector spaces with dimension vector α^{gtr} ; we write $V_{i, n}^\ell$ for the (i, n) -graded piece.

Remark 5.1. We again emphasize that $V_{\bullet, \bullet}, W_{\bullet, \bullet}$ will be endowed with the structure of representations of Q^{gtr} satisfying the relations of A . Our choice of notation for the space $W_{\bullet, \bullet}$ is *not* meant to indicate any relationship to Nakajima's framing vector space $(W_i)_{i \in I}$.

Convention 5.2. We now fix an $N \geq 2$ and set $[a, b] = [0, N]$ in the definitions of $Q^{\text{gtr}}, \alpha^{\text{gtr}}, A$.

Suppose that we choose representations of A in $V_{\bullet, \bullet}, W_{\bullet, \bullet}$; we write $(a^V, e^V) = (a_{h, n}^V, e_{i, n}^V)$, respectively $(a^W, e^W) = (a_{h, n}^W, e_{i, n}^W)$ to denote these two structures. We also write

$$a_n^V = \sum_{h \in H} a_{h, n}^V \quad \text{and} \quad e_n^V = \sum_{i \in I} e_{i, n}^V, \quad \text{and similarly for } W.$$

Notation 5.3. *Given a linear operator L between such graded vector spaces, we sometimes write L_i to mean “the component of the operator acting on the i -graded piece of the domain;” for example, the notation $[(e_1^V)^{-1} \lambda e_1^W]_{s(\bar{h}_0)}$ is used in Equation (5.4) to mean the component of $(e_1^V)^{-1} \lambda e_1^W$ acting at vertex $s(\bar{h}_0)$. We also remind the reader that $s(\bar{h}) = t(h)$, which explains some possibly confusing indices in the proof of Proposition 5.5 below.*

Assumption 5.4. *We assume that the representation $V_{\bullet, \bullet}$ lies in the image of Ind : in other words, the linear operators $e_{i, n}^V$ are invertible for $n \in [0, N - 1]$.*

Consider the vector spaces and maps, graded so $E(V_{\bullet, 0}, W_{\bullet, 1})$ lies in cohomological degree 0,

$$(5.1) \quad L(V_{\bullet, 0}, W_{\bullet, 0}) \xrightarrow{\partial_0} E(V_{\bullet, 0}, W_{\bullet, 1}) \xrightarrow{\partial_1} L(V_{\bullet, 0}, W_{\bullet, 2}),$$

defined as follows: given $\phi \in L(V_{\bullet, 0}, W_{\bullet, 0})$ and $\psi \in E(V_{\bullet, 0}, W_{\bullet, 0})$, we let

$$\partial_0(\phi) = a_0^W \phi - e_0^W \circ \phi \circ (e_1^V)^{-1} a_0^V, \quad \partial_1(\psi) = (\epsilon a_1^W) \psi + e_1^W \circ \psi \circ (e_1^V)^{-1} (\epsilon a_0^V).$$

Proposition 5.5.

- (1) *The kernel of ∂_0 is naturally identified with a subspace of $\text{Hom}_A(V_{\bullet, \bullet}, W_{\bullet, \bullet})$.*
- (2) *The composite $\partial_1 \circ \partial_0$ is zero.*
- (3) *If $[a, b] = [0, 2]$, the cokernel of ∂_1 is naturally identified with $\text{Hom}_A(W_{\bullet, \bullet}, V_{\bullet, \bullet})^*$.*

We note that for assertion (3), we use in a fundamental way that Remark 4.9 applies to $V_{\bullet,\bullet}$.

Proof. If $\partial_0(\phi) = 0$, then we may define a linear map $\Phi_{\bullet} : V_{\bullet,\bullet} \rightarrow W_{\bullet,\bullet}$ by $\Phi_n = e_{n-1}^W \dots e_0^W \circ \phi \circ (e_{n-1}^V \dots e_0^V)^{-1}$. It is immediate from the construction that Φ_{\bullet} commutes with the operators e_n in the obvious sense. Similarly, since $\partial_0(\phi) = 0$ we get that $a_0^W \Phi_0 = \Phi_1 a_0^V$; it is immediate by induction that Φ_{\bullet} is compatible with all operators a in the obvious sense. Thus $\Phi_{\bullet} \in \text{Hom}_A(V_{\bullet,\bullet}, W_{\bullet,\bullet})$. Since $e \in A$ acts invertibly on $V_{\bullet,\bullet}$ in the appropriate range, any such Φ_{\bullet} is determined uniquely by $\Phi_0 = \phi$ by the above construction, proving assertion (1).

For assertion (2), we calculate:

$$(5.2) \quad \partial_1 \partial_0(\phi) = (\epsilon a_1^W) a_0^W \phi - (\epsilon a_1^W) e_0^W \phi (e_1^V)^{-1} a_0^V + e_1^W a_0^W \phi (e_1^V)^{-1} (\epsilon a_0^V) - e_1^W e_0^W \phi (e_1^V)^{-1} a_0^V (e_1^V)^{-1} (\epsilon a_0^V).$$

Now

$$-(\epsilon a_1^W) e_0^W \phi (e_1^V)^{-1} a_0^V + e_1^W a_0^W \phi (e_1^V)^{-1} (\epsilon a_0^V) = -e_1^W (\epsilon a_1^W) \phi (e_1^V)^{-1} a_0^V + e_1^W a_0^W \phi (e_1^V)^{-1} (\epsilon a_0^V) = 0.$$

Thus to prove (2) it suffices to show that

$$(\epsilon a_1^W) a_0^W \phi - e_1^W e_0^W \phi (e_1^V)^{-1} a_0^V (e_1^V)^{-1} (\epsilon a_0^V) = (\epsilon a_1^W) a_0^W \phi - e_1^W e_0^W \phi (e_1^V)^{-1} (e_2^V)^{-1} a_1^V (\epsilon a_0^V) = 0$$

However, $(\epsilon a_1^W) a_0^W = 0 = a_1^V (\epsilon a_0^V)$ is immediate from the preprojective relations.

We now turn to assertion (3). Suppose $\lambda : W^2 \rightarrow V^0$ is an I -graded linear map. We have that $\text{tr}(\lambda \partial_1(\psi)) = 0$ for all $\psi \in E(V^0, W^1)$ if and only if

$$0 = \text{tr}(\lambda(\epsilon a_1^W) \psi) + \text{tr}(\lambda e_1^W \circ \psi \circ (e_1^V)^{-1} (\epsilon a_0^V)) = \text{tr}((\lambda(\epsilon a_1^W) + (e_1^V)^{-1} (\epsilon a_0^V) \lambda e_1^W) \circ \psi)$$

for all ψ , if and only if

$$(5.3) \quad \lambda(\epsilon a_1^W) + (\epsilon a_0^V) (e_1^V)^{-1} \lambda e_1^W = 0.$$

More precisely, this formula “unpacks” as follows. Suppose that $\psi = (\psi_h)_{h \in H}$ and assume given an $h_0 \in H$ with $\psi_h = 0$ for $h \neq h_0$. Then the trace condition reads

$$\text{tr} \left[\lambda_{t(h_0)} (\epsilon a_{h_0,1}^W) \psi_{h_0} + (\epsilon a_{h_0,0}^V) [(e_{s(\bar{h}_0),1}^V)^{-1} \lambda_{s(\bar{h}_0)} (e_{t(h_0),1}^W)] \psi_{h_0} \right] = 0.$$

Since $\psi_{h_0} : V_{s(h_0)}^0 \rightarrow W_{t(h_0)}^1$ is arbitrary, it follows that

$$(5.4) \quad \lambda_{t(h_0)} (\epsilon a_{h_0,1}^W) + (\epsilon a_{h_0,0}^V) [(e_1^V)^{-1} \lambda e_1^W]_{s(\bar{h}_0)} = 0 = \lambda_{t(h_0)} (a_{h_0,1}^W) + (a_{h_0,0}^V) [(e_1^V)^{-1} \lambda e_1^W]_{s(\bar{h}_0)}.$$

By the nondegeneracy of the trace pairing, we obtain:

Lemma 5.6. *The cokernel of ∂_1 is naturally dual to the space of those λ satisfying (5.3).*

We now use Assumption 5.4 and Remark 4.9 to see that $V_{\bullet,\bullet}$ lifts to an $I \times \mathbb{Z}$ -graded $\Pi^0[e]$ -module $\tilde{V}_{\bullet,\bullet}$ with $\dim(V_{i,n}) = \alpha_i$ for all $i \in I$ and $n \in \mathbb{Z}$, in such a way that $\tilde{V}_{\bullet,\bullet+1} \cong \tilde{V}_{\bullet,\bullet}$ via multiplication by e . More precisely, writing $(\tilde{V}_{\bullet,\bullet})_{[i,j]} := \tilde{V}_{\bullet,\bullet \geq i} / \tilde{V}_{\bullet,\bullet \geq j+1}$, we extend λ to a graded linear map $\Lambda_{\bullet} : W_{\bullet,\bullet} \rightarrow \tilde{V}_{\bullet,\bullet}(-2)_{[0,2]} \cong V_{\bullet,\bullet}$ by taking

$$\Lambda_2 = \lambda, \quad \Lambda_1 = -(e_1^V)^{-1} \lambda e_1^W, \quad \Lambda_0 = (e_0^V)^{-1} (e_1^V)^{-1} \lambda e_1^W e_0^W,$$

similarly to our construction of Φ_{\bullet} above. As in our construction of Φ_{\bullet} , it follows from Equation (5.3) that Λ_{\bullet} is indeed a graded A -module homomorphism; and that any graded A -module homomorphism $\Lambda_{\bullet} : W_{\bullet,\bullet} \rightarrow V_{\bullet,\bullet}$ is uniquely determined by $\Lambda_2 = \lambda$ by this construction, completing the proof. \square

Corollary 5.7. *When Q is a Crawley-Boevey quiver and $[a, b] = [0, 2]$, then the complex (5.1) descends to a perfect complex C on $\mathfrak{M} \times \mathfrak{M}$.*

Proof. When $Q = (Q_0)^{\text{CB}}$ is a Crawley-Boevey quiver, we have $\mathbb{G} \cong \mathbb{G}_0 \times \mathbb{G}_m$, where \mathbb{G}_m acts trivially on the stable locus and \mathbb{G}_0 acts freely on the stable locus of $\text{Rep}(\Pi^0(Q), \alpha)$ with quotient \mathfrak{M} . Similarly, $\mathbb{G}^{\text{gtr}} \cong (\mathbb{G}_0)^3 \times \mathbb{G}_m^3$; the subgroup $(\mathbb{G}^{\text{gtr}})_0 = (\mathbb{G}_0)^3 \times \mathbb{G}_m^2 \times \{1\}$ acts freely on $\text{Rep}(A, \alpha^{\text{gtr}})^s$ with quotient \mathfrak{M} . Since the complex defined by (5.1) is $(\mathbb{G}^{\text{gtr}})_0 \times (\mathbb{G}^{\text{gtr}})_0$ -equivariant, it descends to a perfect complex C on $\mathfrak{M} \times \overline{\mathfrak{M}}$. \square

6. PROOFS OF THEOREMS 1.2, 1.3, AND 1.6

Let Q_0 be a quiver with dimension vector \mathbf{v} and framing vector \mathbf{w} , and let $Q = Q_0^{\text{CB}}$ be the Crawley-Boevey quiver associated to Q_0 and \mathbf{w} .

We take $[a, b] = [0, 2]$ in the definitions of Q^{gtr} , etc.

Let $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$ denote the compactification of the quiver variety constructed in Section 4.5. We wish to modify slightly the complex of (5.1) and Corollary 5.7. Thus, we consider the splitting

$$L(V_{\bullet,0}, W_{\bullet,0}) = L(V_{\bullet,0}, W_{\bullet,0})_{I_0} \oplus \mathbb{C} := \left[\bigoplus_{i \in I_0} \text{Hom}(V_{i,0}, W_{i,0}) \right] \oplus \text{Hom}(V_{\infty,0}, W_{\infty,0}).$$

$$\text{We write } \delta_0 = \partial_0|_{L(V_{\bullet,0}, W_{\bullet,0})_{I_0}} : L(V_{\bullet,0}, W_{\bullet,0})_{I_0} \rightarrow E(V_{\bullet,0}, W_{\bullet,1}).$$

Similarly, we consider the splitting

$$L(V_{\bullet,0}, W_{\bullet,2}) = L(V_{\bullet,0}, W_{\bullet,2})_{I_0} \oplus \mathbb{C} := \left[\bigoplus_{i \in I_0} \text{Hom}(V_{i,0}, W_{i,2}) \right] \oplus \text{Hom}(V_{\infty,0}, W_{\infty,2})$$

and write $\delta_1 = \pi \circ \partial_1$ for the composite of ∂_1 followed by the projection

$$\pi : L(V_{\bullet,0}, W_{\bullet,2}) \twoheadrightarrow L(V_{\bullet,0}, W_{\bullet,2})_{I_0}.$$

It is immediate from Corollary 5.7 that we obtain a complex on $\mathfrak{M} \times \overline{\mathfrak{M}}$, namely

$$(6.1) \quad R : \mathcal{L}(V_{\bullet,0}, W_{\bullet,0})_{I_0} \xrightarrow{\delta_0} \mathcal{E}(V_{\bullet,0}, W_{\bullet,1}) \xrightarrow{\delta_1} \mathcal{L}(V_{\bullet,0}, W_{\bullet,2})_{I_0}.$$

Remark 6.1. The complex (6.1) is evidently of the form (2.4).

Theorem 6.2. *For the complex R of (6.1), we have:*

- (1) δ_0 is injective and δ_1 is surjective on each fiber. In particular, $\mathcal{H}^1(R) = 0 = \mathcal{H}^1(R^\vee)$, and $\mathcal{H}^0(R)$ is a vector bundle on $\mathfrak{M} \times \overline{\mathfrak{M}}$.
- (2) the map $\mathbb{C} = \text{Hom}(V_{\infty,0}, W_{\infty,0}) \rightarrow E(V_{\bullet,0}, W_{\bullet,1})$ defines a section s of $\mathcal{H}^0(R)$ whose scheme-theoretic zero locus is the graph Γ of the inclusion $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$.
- (3) $\text{rk}(R) = \dim(\overline{\mathfrak{M}})$.

Proof. (1) By Proposition 5.5, when $V_{\bullet,\bullet}$ and $W_{\bullet,\bullet}$ are stable, $\ker(\partial_0)$ is zero or consists of multiples of the identity endomorphism of $V_{\bullet,\bullet} \cong W_{\bullet,\bullet}$; in either case, we have $\ker(\partial_0) \cap L(V_{\bullet,0}, W_{\bullet,0})_{I_0} = 0$. Thus δ_0 is injective on each fiber.

Similarly either $\text{coker}(\partial_1)$ is zero, or else $V_{\bullet,\bullet} \cong W_{\bullet,\bullet}$ and $\text{coker}(\partial_1) \cong \text{Hom}(W_{\bullet,\bullet}, V_{\bullet,\bullet})^* \cong \mathbb{C}$ by stability of $V_{\bullet,\bullet}$ and $W_{\bullet,\bullet}$; in the latter case, since $\text{im}(\partial_1)$ has codimension 1, its projection on $L(V_{\bullet,0}, W_{\bullet,2})_{I_0}$ must be surjective: otherwise $\text{im}(\partial_1) \cap \text{Hom}(V_{\infty,0}, W_{\infty,2}) \neq 0$, but every nonzero element of its dual $\text{Hom}(W_{\infty,0}, V_{\infty,0})$ is nonzero at the vertex ∞ . We conclude that δ_1 is surjective on each fiber, concluding the proof of assertion (1).

(2) By Proposition 5.5, the cohomologies $H^1(C)$ and $H^1(C^\vee)$ are supported set-theoretically on the graph Γ of the inclusion $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$. It follows that the set-theoretic zero locus of the section s of assertion (2) is Γ . Thus, to prove the stronger scheme-theoretic assertion, we may restrict R to

$\mathfrak{M} \times \mathfrak{M}$. Supposing, then, that both $e_{i,n}^V$ and $e_{i,n}^W$ act invertibly for $n = 0, 1$, we write V, W for the corresponding representations of Π^0 . We note that the complex (5.1) is then identified with

$$L(V, W) \xrightarrow{\partial_0} E(V, W) \xrightarrow{\partial_1} L(V, W),$$

$$\partial_0(\phi) = a_0^W \phi - \phi a_0^V, \quad \partial_1(\psi) = (\epsilon a_1^W) \psi + \psi (\epsilon a_0^V).$$

The assertion (2) now follows immediately from Proposition 3.5.

(3) The rank assertion is immediate by direct calculation as in [Nak1]. \square

Proof of Theorem 1.2. Let $d = \dim(\mathfrak{M})$. By Theorem 6.2 and Remark 6.1, the hypotheses of Corollary 2.5 are satisfied. Theorem 1.2 follows. \square

Proof of Theorem 1.3. By Theorem 7.3.5 of [Nak2], $H^*(\mathfrak{M}, \mathbb{Z})$ is known to be free abelian and concentrated in even degrees. By the universal coefficient theorem, it follows that for any graded ring $E^*(\text{pt})$, $H^*(\mathfrak{M}, \mathbb{Z}) \otimes_{\mathbb{Z}} E^*(\text{pt}) = H^*(\mathfrak{M}, E^*(\text{pt}))$ and $H^*(B\mathbb{G}, \mathbb{Z}) \otimes_{\mathbb{Z}} E^*(\text{pt}) = H^*(B\mathbb{G}, E^*(\text{pt}))$.

The Atiyah-Hirzebruch spectral sequence for a cohomology theory E and space X has E_2 -page $E_2^{p,q} = H^p(X, E^q(\text{pt})) \implies E^{p+q}(X)$. By the previous paragraph, if $E^*(\text{pt})$ is evenly graded the spectral sequence degenerates at E_2 for both $E^*(\mathfrak{M})$ and $E^*(B\mathbb{G})$. Assertion (1) of the theorem thus follows from Theorem 1.2.

To prove (2), we observe that all the ingredients of the proof of Proposition 2.4 hold in any complex-oriented cohomology theory E . In particular, there is a Gysin map for proper morphisms and one can calculate f^* via pull-cup-with-graph-push; that $[\Gamma] = c_d(R)$ and Chern classes of R depend polynomially on the Chern classes of the tautological bundles follow from explicit formulas as in Lemmas 2.1 and 2.3 of [Hu]. It remains to see that $E^*(\overline{\mathfrak{M}}) \rightarrow E^*(\mathfrak{M})$ is surjective; however, the natural map $\mathfrak{M} \rightarrow B\mathbb{G}$ factors through $\overline{\mathfrak{M}} \rightarrow B\mathbb{G}$ defined via projection of \mathbb{G}^{gtr} on any factor \mathbb{G} , and surjectivity of $E^*(\overline{\mathfrak{M}}) \rightarrow E^*(\mathfrak{M})$ follows from that of $E^*(B\mathbb{G}) \rightarrow E^*(\mathfrak{M})$. \square

Proof of Theorem 1.6. We note that assertion (1) is immediate from assertion (2).

In light of Remark 6.1, we will use the notation of Proposition 2.4 for the complex R . The Koszul complex associated to the complex R and section s of $\mathcal{H} = \mathcal{H}^0(R)$ of Theorem 6.2 provides a resolution (Section B.3.4 of [Ful]) of \mathcal{O}_{Γ} ,

$$(6.2) \quad \left[\bigwedge^d \mathcal{H}^* \rightarrow \cdots \rightarrow \bigwedge^2 \mathcal{H}^* \rightarrow \mathcal{H}^* \rightarrow \mathcal{O}_{\mathfrak{M} \times \overline{\mathfrak{M}}} \right] \simeq \mathcal{O}_{\Gamma}.$$

For each k , consider the k th tensor power $T^k(R)$ of the complex R : it is a differential graded vector bundle whose terms are tensor products of \mathcal{E}_j^ℓ s and \mathcal{F}_j^ℓ s. The symmetric group S_k naturally acts on $T^k(R)$ with the usual $\mathbb{Z}/2\mathbb{Z}$ -graded sign conventions; we write $\bigwedge^k(R) = T^k(R)^{S_k, \text{sgn}}$, the sign-isotypic part of $T^k(R)$. Both operations $T^k(-)$ and $(-)^{S_k, \text{sgn}}$ preserve quasi-isomorphism, hence $\bigwedge^k(R) \simeq \bigwedge^k(\mathcal{H})$. The Koszul complex thus writes \mathcal{O}_{Γ} as an iterated cone on the complexes $\bigwedge^k(R)^\vee$.

We remark that, viewing $\mathcal{E}^\bullet := \oplus_j \mathcal{E}_j^\bullet$ and $\mathcal{F}^\bullet := \oplus_j \mathcal{F}_j^\bullet$ as $\mathbb{Z}/2$ -graded vector bundles, we find that $\bigwedge^k(R)$ is a direct summand of $\bigwedge^k(\mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet)$ in a canonical way. Furthermore, following the work of [BR]² it is known that $\bigwedge^k(\mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet)$ is an iterated extension of tensor products of Schur functors applied to the $\mathbb{Z}/2$ -graded vector bundles \mathcal{E}^\bullet and \mathcal{F}^\bullet (see Corollary 1.2 of [EW] and the discussion preceeding it for more details). Moreover, the expression for $\bigwedge^k(\mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet)$ as an iterated extension of $S_\lambda(\mathcal{E}^\bullet)$ and $S_\lambda(\mathcal{F}^\bullet)$ is compatible with the expression for $\bigwedge^k(R)$ as a direct summand of $\bigwedge^k(\mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet)$: in particular, $\bigwedge^k(R)$ is an iterated cone on external tensor products of the objects $S_\lambda(\mathcal{E}_j^\ell)$, $S_\lambda(\mathcal{F}_j^\ell)$ that are obtained by applying Schur functors to the various \mathcal{E}_j^ℓ and \mathcal{F}_j^ℓ .

²We thank J. Weyman for help with references.

Suppose \mathcal{G} is a coherent complex on $\overline{\mathfrak{M}}$. For any external tensor product $S_\lambda(\mathcal{E}_j^\ell)^\vee \boxtimes N$, we have

$$\mathbb{R}(p_{\mathfrak{M}})_*((S_\lambda(\mathcal{E}^\ell)^\vee \boxtimes N) \otimes (p_{\mathfrak{M}})^*\mathcal{G}) \simeq S_\lambda(\mathcal{E}_j^\ell)^\vee \otimes U^\bullet$$

for some bounded complex U^\bullet of finite-dimensional vector spaces. Using (6.2) and the conclusion of the previous paragraph, we find that $\mathcal{G}|_{\mathfrak{M}}$ lies in the subcategory of $D_{\text{coh}}(\mathfrak{M})$ that is generated, under the operations (i)-(iii) of assertion (2) of Theorem 1.6, by the $S_\lambda(\mathcal{E}_j^\ell)^\vee$, where the Schur functors that appear are exactly those used in writing all the $\bigwedge^k(\mathcal{H})$ as above. \square

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